Available online at www.sciencedirect.com

SCIENCE DIRECT

Journal of Sound and Vibration 292 (2006) 474–487

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

A perturbation approach for the eigenfrequency analysis of separable Kirchhoff thin-plate problems on a rectangle

Matthew P. Coleman*, Laura A. McSweeney

Department of Mathematics and Computer Science, Fairfield University, Fairfield, CT 06824-5195, USA

Received 24 August 2004; accepted 6 August 2005
Available online 21 October 2005

Abstract

We introduce a method for the estimation of the eigenfrequencies for separable Kirchhoff thin-plate problems on a rectangle, particularly those involving the free boundary condition. We believe that this paper is the first to give such a treatment for the low end of the spectrum. The method is an adaptation of an asymptotic/perturbation method used for the treatment of various beam problems and, in the setting of the plate, is shown to be a generalization and refinement of the asymptotic methods of Bolotin and Keller and Rubinow (the wave propagation method). We compare our results with those of our own Legendre-tau approximation and, where available, with numerical results extant in the literature. Excellent agreement is found in each case.

1. Introduction

We consider the very important model of the Kirchhoff thin plate on a rectangle, subject to one of the four naturally occurring energy-conserving boundary conditions along each edge. Thus, we have the PDE

$$w_{tt}(x,y,t)+\nabla^4 w(x,y,t)=0,\quad (x,y)\in\Omega=(0,a)\times(0,b),\quad t>0, \tag{1}$$

where ∇^4 is the biharmonic operator, along with one of the following boundary configurations along each vertical edge $x=$ constant and each horizontal edge $y=$ constant.

Boundary conditions along edges $x=$ constant:

$$\text{C (clamped)}\quad w=w_x=0,\quad t>0, \tag{2}$$

$$\text{S (simply supported)}\quad w=w_{xx}+\nu w_{yy}=0\quad (\Rightarrow w_{xx}=0),\quad t>0, \tag{3}$$

$$\text{R (roller-supported)}\quad w_x=w_{xxx}+(2-\nu)w_{xyy}=0\quad (\Rightarrow w_{xxx}=0),\quad t>0, \tag{4}$$

$$\text{F (free)}\quad w_{xx}+\nu w_{yy}=w_{xxx}+(2-\nu)w_{xyy}=0,\quad t>0. \tag{5}$$

*Corresponding author. Tel.: +1 203 254 4000; fax: +1 203 254 4163.
E-mail addresses: mcoleman@mail.fairfield.edu (M.P. Coleman), lmcsweeney@mail.fairfield.edu (L.A. McSweeney).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2005.08.013

Nomenclature			
		λ	vibration frequency
		C	clamped boundary condition
u_x	$\partial u/\partial x$	S	simply supported boundary condition
v	Poisson's ratio	R	roller-supported boundary condition
ε	perturbation variable	F	free boundary condition
P_n	Legendre polynomial of degree n		

Boundary conditions along edges y = constant:

$$\text{C} \quad w = w_y = 0, \quad t > 0, \tag{6}$$

$$\text{S} \quad w = w_{yy} = 0, \quad t > 0, \tag{7}$$

$$\text{R} \quad w_y = w_{yyy} = 0, \quad t > 0, \tag{8}$$

$$\text{F} \quad w_{yy} + vw_{xx} = w_{yyy} + (2 - v)w_{xxy} = 0, \quad t > 0. \tag{9}$$

Note that in Eq. (3), we have that $w(t, y) = 0 \Rightarrow w_{yy}(t, y) = 0$; similarly, in Eq. (4), $w_x(t, y) = 0 \Rightarrow w_{xyy}(t, y) = 0$. Of course, similar reasoning leads to the second equation in each of Eqs. (7) and (8). In each case, v is the constant Poisson's ratio, $0 < v < \frac{1}{2}$, and we have used the notation $w_t = \partial w/\partial t$, etc. We identify the various boundary configurations using Leissa's convention [1]. For example, the sequence S–C–F–C indicates that the edges $x = 0$, $y = 0$, $x = a$ and $y = b$ are simply supported, clamped, free and clamped, respectively, as shown in Fig. 1.

Of course, crucial to the understanding of the behavior of any vibration problem is an accurate computation of the vibration spectrum. It is well known that, if each edge of the plate is either simply supported or roller supported, then each frequency is just the square of the corresponding frequency for the clamped membrane. In the case at hand, as we shall see, two opposite sides will be such that each is either simply supported or roller-supported, while at least one of the other edges is clamped or free. Here, the frequencies can be computed to any desired degree of accuracy, using numerical methods. In fact, nowadays, there are various commercial software packages, like MSC/NASTRAN and FEMLAB, which will solve these problems.

Instead, we would like to extend the ideas used in Refs. [2,3]. There, a perturbation method was treated as an add-on to the asymptotic wave propagation method (WPM) of Keller and Rubinow [4,5], the perturbations being necessary to ensure accuracy at the low end of the spectrum, corresponding to the highest energies of vibration. Here, we extend that method to those plate problems for which it is applicable; however, we treat it as the stand-alone method that it is, with WPM, as well as Bolotin's asymptotic method [6], appearing as special cases.

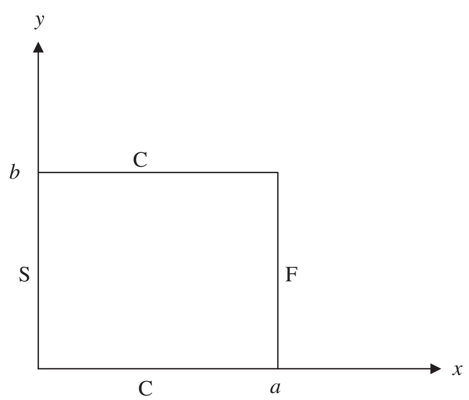

Fig. 1. S–C–F–C plate.

So, in Section 2, we look at the boundary configurations for which the approach is applicable. Here, we choose three representative configurations, while we lay out the method for each case in Section 3. Section 4 sets up the actual equations to be solved and how to solve them and also briefly describes the Legendre-tau spectral method, which we apply to the problem for the sake of comparison. The results are presented in Section 5, along with a comparison of the Legendre-tau results and other numerical results from the literature.

Now, in Section 3, the third configuration was chosen so as to highlight not only the power, but also the limitation, of the method. Therefore, in Section 6, we show how to circumvent this one limitation.

While providing an elegant solution to these plate problems, this paper's important contributions are (1) that it is the first step in generalizing a successful one-dimensional method to the solution of problems in higher dimensions; (2) that it supplants two well-known asymptotic methods for solving these problems, as it contains each as a special case; and (3) that this method, if generalized to arbitrary rectangular plate problems (and, thus, we assume, to shallow shell problems on rectangular domains), and made rigorous, may in fact become part of the arsenal of routines used by various commercial software packages.

2. Separation and applicable boundary configurations

First we separate time from the other variables. Letting

$$w(x, y, t) = e^{i\lambda t}\phi(x, y), \tag{10}$$

the PDE becomes

$$\nabla^4\phi - \lambda^2\phi = 0 \quad (x, y) \in \Omega, \tag{11}$$

while the new BCs are identical to Eqs. (2)–(9), but with w replaced by ϕ (and, of course, here there is no need for the statement $t>0$).

Now, the problem is to determine which plate boundary configurations will allow us to apply the method from Refs. [2,3]. It is easy to see that the method was successful for beam problems because a complete "general solution" could be found. Thus, the same should be true in the case of the plate and, as we shall see, this will be the case exactly when the plate problem is separable. Indeed, when the problem is not separable we find that whatever linear combination of solutions we begin with, application of the boundary conditions leads to an overdetermined system which has only the trivial solution.

So, we separate the variables,

$$\phi(x, y) = X(x)Y(y)$$

and the PDE (11) becomes

$$\frac{X^{(4)}}{X} + \frac{2X''Y''}{XY} + \frac{Y^{(4)}}{Y} - \lambda^2 = 0.$$

Thus, it is separable exactly when

$$X'' = -\alpha^2 X \quad \text{or} \quad Y'' = -\alpha^2 Y$$

for some real constant α. Aside from the degenerate case $\alpha = 0$, one of these conditions will apply exactly when a pair of opposite sides is such that each is simply supported or roller-supported. Neglecting those cases which are trivial (S or R along each edge), we have the following configurations to consider:

S−C−S−C, S−C−S−S, S−C−S−R, S−C−S−F, S−S−S−F, S−R−S−F, S−F−S−F,

S−C−R−C, S−C−R−S, S−C−R−R, S−C−R−F, S−S−R−F, S−R−R−F, S−F−R−F,

R−C−R−C, R−C−R−S, R−C−R−R, R−C−R−F, R−S−R−F, R−R−R−F, R−F−R−F.

Rather than treating all 21, we wish to choose three representative cases. We may cull the list by realizing that a number of configurations are impractical, physically (e.g. R–C–R–C). Further, the C, S and R boundary conditions for the plate are essentially the same as for the beam (to be more precise, see the end of Section 5), while the F condition is more complex, due to the presence of the Poisson's ratio term.

We have decided to look at the three cases S–C–S–C, S–C–S–F and S–F–S–F. The first is the "easiest" and will allow us clearly to illustrate the method. In addition, the first and third have been treated elsewhere in the literature (via numerical methods), so there are results, at least for the lowest few frequencies, with which we can compare ours. Finally, in our experience, the asymmetry of cases involving C-opposite-F tends to make that problem the most difficult to deal with.

3. Application of the perturbation method

We begin by writing $\lambda^2 = k^4$ and forming the following "general solution" of Eq. (11):

$$\begin{aligned}\phi(x,y) = {} & A_1 e^{-i(k_1 x + k_2 y)} + A_2 e^{i(k_1 x - k_2 y)} + A_3 e^{i(k_1 x + k_2 y)} + A_4 e^{i(-k_1 x + k_2 y)} \\ & + B_1 e^{-\sqrt{k_1^2+2k_2^2}x - ik_2 y} + B_2 e^{-\sqrt{k_1^2+2k_2^2}(a-x) - ik_2 y} + B_3 e^{-\sqrt{k_1^2+2k_2^2}x + ik_2 y} \\ & + B_4 e^{-\sqrt{k_1^2+2k_2^2}(a-x) + ik_2 y} + C_1 e^{-ik_1 x - \sqrt{2k_1^2+k_2^2}y} + C_2 e^{-ik_1 x - \sqrt{2k_1^2+k_2^2}(b-y)} \\ & + C_3 e^{ik_1 x - \sqrt{2k_1^2+k_2^2}y} + C_4 e^{ik_1 x - \sqrt{2k_1^2+k_2^2}(b-y)} + D_1 e^{-k_1 x - k_2 y} \\ & + D_2 e^{-k_1(a-x) - k_2 y} + D_3 e^{-k_1 x - k_2(b-y)} + D_4 e^{-k_1(a-x) - k_2(b-y)}. \end{aligned} \tag{12}$$

Here, $k^2 = k_1^2 + k_2^2$, and we search for values of k_1 and k_2 for which Eq. (12) survives the boundary conditions.

3.1. Case 1: S–C–S–C

We apply boundary condition (3) to the edges $x = 0$ and a and boundary condition (6) to the edges $y = 0$ and b. For example, along $x = 0$,

$$\begin{aligned}\phi(0,y) = 0 = {} & e^{-ik_2 y}\left[A_1 + A_2 + B_1 + B_2 e^{-\sqrt{k_1^2+2k_2^2}a}\right] \\ & + e^{ik_2 y}\left[A_3 + A_4 + B_3 + B_4 e^{-\sqrt{k_1^2+2k_2^2}a}\right] \\ & + e^{-\sqrt{2k_1^2+k_2^2}y}[C_1 + C_3] + e^{-\sqrt{2k_1^2+k_2^2}(b-y)}[C_2 + C_4] \\ & + e^{-k_2 y}[D_1 + D_2 e^{-k_1 a}] + e^{-k_2(b-y)}[D_3 + D_4 e^{-k_1 a}], \quad 0 < y < b, \end{aligned}$$

implying that we must have

$$A_1 + A_2 + B_1 + B_2 e^{-\sqrt{k_1^2+2k_2^2}a} = 0,$$

$$A_3 + A_4 + B_3 + B_4 e^{-\sqrt{k_1^2+2k_2^2}a} = 0,$$

$$C_1 + C_3 = 0, \quad C_2 + C_4 = 0,$$

$$D_1 + D_2 e^{-k_1 a} = 0, \quad D_3 + D_4 e^{-k_1 a} = 0.$$

We proceed similarly for the remaining seven BCs. It follows immediately, from the fact that the edge $y = 0$ is clamped, that

$$B_1 = B_2 = B_3 = B_4 = 0.$$

Then, the two S conditions imply that

$$k_1 = \frac{n\pi}{a}, \quad n = 1, 2, 3, \ldots$$

and also that

$$A_2 = -A_1, \quad A_4 = -A_3, \quad C_4 = -C_2, \quad C_3 = -C_1$$

and

$$D_1 = D_2 = D_3 = D_4 = 0.$$

So the system of 16 equations in 16 unknowns very quickly reduces to the 4×4 system

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & \varepsilon \\ \mathrm{i}k_2 & -\mathrm{i}k_2 & \sqrt{2k_1^2+k_2^2} & -\sqrt{2k_1^2+k_2^2}\varepsilon \\ \mathrm{e}^{-\mathrm{i}k_2 b} & \mathrm{e}^{\mathrm{i}k_2 b} & \varepsilon & 1 \\ \mathrm{i}k_2\mathrm{e}^{-\mathrm{i}k_2 b} & -\mathrm{i}k_2\mathrm{e}^{\mathrm{i}k_2 b} & \sqrt{2k_1^2+k_2^2}\varepsilon & -\sqrt{2k_1^2+k_2^2} \end{bmatrix}}_{M} \begin{bmatrix} A_1 \\ A_4 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where $\varepsilon = \mathrm{e}^{-\sqrt{2k_1^2+k_2^2}b}$ and $k_1 = n\pi/a$. The system has nontrivial solutions if and only if

$$0 = \det M = 2[\mathrm{e}^{\mathrm{i}k_2 b}z - \mathrm{e}^{-\mathrm{i}k_2 b}\overline{z}] + \varepsilon 8\mathrm{i}k_2\sqrt{2k_1^2+k_2^2} + 2\varepsilon^2[\mathrm{e}^{-\mathrm{i}k_2 b}z - \mathrm{e}^{\mathrm{i}k_2 b}\overline{z}], \tag{13}$$

where

$$z = z(k_1, k_2) = k_1^2 - \mathrm{i}k_2\sqrt{2k_1^2+k_2^2}.$$

For this complex transcendental equation, or the corresponding real system of two transcendental equations, we have been unable to obtain acceptable solutions using numerical routines.

Instead, realizing that ε is indeed quite small when $b/a \sim 1$, even when $n = 1$, we treat ε as a *perturbation variable*. First let us write

$$k_2 = x, \quad M = M_\varepsilon \quad \text{and} \quad k_1 = r_n = \frac{n\pi}{a}, \quad n \in \mathbb{Z}^+$$

(where we have abused the notation, as this x has nothing to do with the original Cartesian variable, of course). So we must solve

$$\begin{aligned} 0 &= \tfrac{1}{2}\det M_\varepsilon \\ &= \mathrm{e}^{\mathrm{i}bx}z - \mathrm{e}^{-\mathrm{i}bx}\overline{z} + \varepsilon 4\mathrm{i}x\sqrt{2r_n^2+x^2} + \varepsilon^2(\mathrm{e}^{-\mathrm{i}bx}z - \mathrm{e}^{\mathrm{i}bx}\overline{z}) \end{aligned}$$

with

$$z = z(r_n, x).$$

Next, we expand x as

$$x = x_0 + x_1\varepsilon + x_2\varepsilon^2 + \cdots.$$

In principle, we may go out to ε^m for any $m \in \mathbb{Z}^+$. However, as we shall see, we need only consider ε^0 and ε^1. So, we write

$$x = x_0 + x_1\varepsilon + O(\varepsilon^2),$$

from which we also have

$$\mathrm{e}^{\pm \mathrm{i}bx} = \mathrm{e}^{\pm \mathrm{i}bx_0}[1 \pm \mathrm{i}bx_1\varepsilon] + O(\varepsilon^2),$$

$$\sqrt{2r_n^2+x^2} = \sqrt{2r_n^2+x_0^2} + \frac{x_0x_1}{\sqrt{2r_n^2+x_0^2}}\varepsilon + O(\varepsilon^2),$$

etc. After much computation, we arrive at

$$\tfrac{1}{2}\det M_\varepsilon = \mathrm{e}^{\mathrm{i}bx_0}z_0 - \mathrm{e}^{-\mathrm{i}bx_0}\overline{z_0} + \varepsilon\left\{[\mathrm{e}^{\mathrm{i}bx_0}w_0 - \mathrm{e}^{-\mathrm{i}bx_0}\overline{w_0}]x_1 + 4\mathrm{i}x_0\sqrt{2r_n^2+x_0^2}\right\} + O(\varepsilon^2),$$

where $z_0 = z(r_n, x_0)$ and

$$w_0 = w(r_n, x_0) = bx_0\sqrt{2r_n^2 + x_0^2} + \mathrm{i}\left[br_n^2 - \sqrt{2r_n^2 + x_0^2} - \frac{x_0^2}{\sqrt{2r_n^2 + x_0^2}}\right].$$

Now, the coefficient of each power of ε must be zero. So we solve for x_0 from the 0th-order approximation

$$\mathrm{e}^{\mathrm{i}bx_0} z_0 - \mathrm{e}^{-\mathrm{i}bx_0}\overline{z_0} = 0. \tag{14}$$

Then, in turn, we solve for x_1 in terms of x_0 from the coefficient of ε^1, the result being

$$x_1 = -\frac{2x_0\sqrt{2r_n^2 + x_0^2}}{\mathrm{Im}(\mathrm{e}^{\mathrm{i}bx_0} w_0)}.$$

Again, we could proceed to do the same thing for higher powers of ε. However, instead, we begin with the approximation

$$x^{(1)} = x_0 + \varepsilon_0 x_1,$$

where the obvious choice for ε is

$$\varepsilon = \varepsilon_0 = \mathrm{e}^{-\sqrt{2r_n^2 + x_0^2}\,b}.$$

Next, we may improve $x^{(1)}$ by updating our choice of ε. So, let

$$\varepsilon = \varepsilon_1 = \mathrm{e}^{-\sqrt{2r_n^2 + x^{(1)2}}\,b} = \mathrm{e}^{-\sqrt{2r_n^2 + (x_0 + \varepsilon_0 x_1)^2}\,b}$$

and our improved estimate is

$$x^{(2)} = x_0 + \varepsilon_1 x_1.$$

We may continue the process as needed. In general, we have

$$\varepsilon_j = \mathrm{e}^{-\sqrt{2r_n^2 + x^{(j)2}}\,b} = \mathrm{e}^{-\sqrt{2r_n^2 + (x_0 + \varepsilon_{j-1} x_1)^2}\,b}$$

with improved estimate

$$x^{(j+1)} = x_0 + \varepsilon_j x_1.$$

3.2. Case 2: S–C–S–F

Here, again, we form the "general solution" (12) and apply the boundary conditions. Again, it follows almost immediately that

$$k_1 = r_n = \frac{n\pi}{a}, \quad n \in \mathbb{Z}^+,$$

$$A_2 = -A_1, \quad A_3 = -A_4,$$

$$B_1 = B_2 = B_3 = B_4 = 0,$$

$$C_3 = -C_1, \quad C_4 = -C_2,$$

$$D_1 = D_2 = D_3 = D_4 = 0. \tag{15}$$

Our final matrix is

$$M_\varepsilon = \begin{bmatrix} 1 & 1 & 1 & \varepsilon \\ \mathrm{i}x & -\mathrm{i}x & \sqrt{2r_n^2+x^2} & -\sqrt{2r_n^2+x^2}\varepsilon \\ f_1\mathrm{e}^{-\mathrm{i}bx} & f_1\mathrm{e}^{\mathrm{i}bx} & -f_2\varepsilon & -f_2 \\ \mathrm{i}xf_2\mathrm{e}^{-\mathrm{i}bx} & -\mathrm{i}xf_2\mathrm{e}^{\mathrm{i}bx} & -\sqrt{2r_n^2+x^2}f_1\varepsilon & \sqrt{2r_n^2+x^2}f_1 \end{bmatrix},$$

where

$$f_1 = f_1(r_n, x) = vr_n^2 + x^2,$$

$$f_2 = f_2(r_n, x) = (2-v)r_n^2 + x^2 \tag{16}$$

and again,

$$\varepsilon = \mathrm{e}^{-\sqrt{2r_n^2+x^2}b}.$$

Then,

$$\det M_\varepsilon = \mathrm{e}^{\mathrm{i}bx}z - \mathrm{e}^{-\mathrm{i}bx}\overline{z} + \varepsilon 8\mathrm{i}xf_1f_2\sqrt{2r_n^2+x^2} + O(\varepsilon^2)$$

with

$$z = z(r_n, x) = x^2f_2^2 - (2r_n^2+x^2)f_1^2 + \mathrm{i}x\sqrt{2r_n^2+x^2}(f_1^2+f_2^2). \tag{17}$$

Letting $x = x_0 + \varepsilon x_1$, and after much computation, we have

$$\det M_\varepsilon = \mathrm{e}^{\mathrm{i}bx_0}z_0 - \mathrm{e}^{-\mathrm{i}bx_0}\overline{z_0} + \varepsilon\left\{[\mathrm{e}^{\mathrm{i}bx_0}w_0 - \mathrm{e}^{-\mathrm{i}bx_0}\overline{w_0}]x_1 + 8\mathrm{i}x_0\sqrt{2r_n^2+x_0^2}f_1(r_n,x_0)f_2(r_n,x_0)\right\} + O(\varepsilon^2). \tag{18}$$

Here, $z_0 = z(r_n, x_0)$ and

$$w_0 = w(r_n, x_0) = x_0\left\{8(1-2v)r_n^2(r_n^2+x_0^2) - 2x_0^4 - 2b\sqrt{2r_n^2+x_0^2}[x_0^4 + 2r_n^2x_0^2 + (2-2v+v^2)r_n^4]\right\}$$
$$+\mathrm{i}\left\{2(1-2v)br_n^2x_0^2(2r_n^2+x_0^2) + 2\sqrt{2r_n^2+x_0^2}[5x_0^4 + 6r_n^2x_0^2 + (2-2v+v^2)r_n^4] + \frac{2x_0^2}{\sqrt{2r_n^2+x_0^2}}[x_0^4 + 2r_n^2x_0^2 + (2-2v+v^2)r_n^4]\right\}.$$

3.3. Case 3: S–F–S–F

Once more, we apply the appropriate BCs to the general solution (12). Here it takes much more work to show that $B_1 = B_2 = B_3 = B_4 = 0$ and $D_1 = D_2 = D_3 = D_4 = 0$, whence Eq. (15) again follows.

Now the coefficient matrix is

$$M_\varepsilon = \begin{bmatrix} f_1 & f_1 & -f_2 & -f_2\varepsilon \\ \mathrm{i}xf_2 & -\mathrm{i}xf_2 & -\sqrt{2r_n^2+x^2}f_1 & \sqrt{2r_n^2+x^2}f_1\varepsilon \\ f_1\mathrm{e}^{-\mathrm{i}bx} & f_1\mathrm{e}^{\mathrm{i}bx} & -f_2\varepsilon & -f_2 \\ \mathrm{i}xf_2\mathrm{e}^{-\mathrm{i}bx} & -\mathrm{i}xf_2\mathrm{e}^{\mathrm{i}bx} & -\sqrt{2r_n^2+x^2}f_1\varepsilon & \sqrt{2r_n^2+x^2}f_1 \end{bmatrix}$$

with $f_1 = f_1(r_n, x)$ and $f_2 = f_2(r_n, x)$ again given by Eq. (16), and

$$\det M_\varepsilon = \mathrm{e}^{\mathrm{i}bx} z - \mathrm{e}^{-\mathrm{i}bx}\overline{z} + \varepsilon 8\mathrm{i}x f_1^2 f_2^2 \sqrt{2r_n^2 + x^2} + O(\varepsilon^2).$$

Here,

$$z = z(r_n, x) = (2r_n^2 + x^2) f_1^4 - x^2 f_2^4 - 2\mathrm{i}x\sqrt{2r_n^2 + x^2} f_1^2 f_2^2. \tag{19}$$

Letting $x = x_0 + \varepsilon x_1$, and after *much* computation, we arrive at

$$\begin{aligned}\det M_\varepsilon &= \mathrm{e}^{\mathrm{i}bx_0} z_0 - \mathrm{e}^{-\mathrm{i}bx_0}\overline{z_0} \\ &+ \varepsilon\left\{\mathrm{i}[\mathrm{e}^{\mathrm{i}bx_0} w_0 - \mathrm{e}^{-\mathrm{i}bx_0}\overline{w_0}]x_1 + 8\mathrm{i}x_0\sqrt{2r_n^2 + x_0^2} f_1^2(r_n, x_0) f_2^2(r_n, x_0)\right\} + O(\varepsilon^2),\end{aligned} \tag{20}$$

where $z_0 = z(r_n, x_0)$ and

$$\begin{aligned}w_0 = w(r_n, x_0) &= b(2r_n^2 + x_0^2) f_1^4(r_n, x_0) - b x_0^2 f_2^4(r_n, x_0) \\ &- 2\sqrt{2r_n^2 + x_0^2} f_1^2(r_n, x_0) f_2(r_n, x_0)[f_2(r_n, x_0) + 4x_0^2] \\ &- \frac{2x_0^2 f_1(r_n, x_0) f_2^2(r_n, x_0)}{\sqrt{2r_n^2 + x_0^2}}[f_1(r_n, x_0) + 4(2r_n^2 + x_0^2)] \\ &+ 2\mathrm{i}\Big\{x_0 f_1^3(r_n, x_0)[f_1(r_n, x_0) + 4(2r_n^2 + x_0^2)] \\ &- x_0 f_2^3(r_n, x_0)[f_2(r_n, x_0) + 4x_0^2] + b x_0\sqrt{2r_n^2 + x_0^2} f_1^2(r_n, x_0) f_2^2(r_n, x_0)\Big\}.\end{aligned}$$

4. Implementation

In each case, our baseline estimates come from the 0th-order approximation

$$\det M_0 = 0.$$

And in each case—indeed, for every such separable plate problem—it can be shown that this equation is equivalent to the corresponding WPM and Bolotin approximations.

4.1. Case 1: S–C–S–C

We begin by solving Eq. (14). We have

$$0 = \mathrm{e}^{\mathrm{i}bx_0}\left(\frac{n^2\pi^2}{a^2} - \mathrm{i}x_0\sqrt{2\frac{n^2\pi^2}{a^2} + x_0^2}\right) - \mathrm{e}^{-\mathrm{i}bx_0}\left(\frac{n^2\pi^2}{a^2} + \mathrm{i}x_0\sqrt{2\frac{n^2\pi^2}{a^2} + x_0^2}\right)$$

or

$$2bx_0 + 2\pi m = \arg\left[\frac{\dfrac{n^2\pi^2}{a^2} + \mathrm{i}x_0\sqrt{2\dfrac{n^2\pi^2}{a^2} + x_0^2}}{\dfrac{n^2\pi^2}{a^2} - \mathrm{i}x_0\sqrt{2\dfrac{n^2\pi^2}{a^2} + x_0^2}}\right]$$

$$
\begin{aligned}
&= \arg\left[\frac{\sqrt{2\dfrac{n^2\pi^2}{a^2}+x_0^2}-\mathrm{i}x_0}{\sqrt{2\dfrac{n^2\pi^2}{a^2}+x_0^2}+\mathrm{i}x_0}\right]^2\\
&= 2\arg\left[\frac{\sqrt{2\dfrac{n^2\pi^2}{a^2}+x_0^2}-\mathrm{i}x_0}{\sqrt{2\dfrac{n^2\pi^2}{a^2}+x_0^2}+\mathrm{i}x_0}\,\frac{\sqrt{2\dfrac{n^2\pi^2}{a^2}+x_0^2}-\mathrm{i}x_0}{\sqrt{2\dfrac{n^2\pi^2}{a^2}+x_0^2}-\mathrm{i}x_0}\right]\\
&= 2\arg\left[\frac{\left(\sqrt{2\dfrac{n^2\pi^2}{a^2}+x_0^2}-\mathrm{i}x_0\right)^2}{2\dfrac{n^2\pi^2}{a^2}+2x_0^2}\right]\\
&= 2\arg\left(\sqrt{2\frac{n^2\pi^2}{a^2}+x_0^2}-\mathrm{i}x_0\right)^2\\
&= 4\arg\left(\sqrt{2\frac{n^2\pi^2}{a^2}+x_0^2}-\mathrm{i}x_0\right)\\
&= -4\tan^{-1}\left(\frac{x_0}{\sqrt{2\dfrac{n^2\pi^2}{a^2}+x_0^2}}\right).
\end{aligned}
$$

Thus we must solve

$$
F(x_0,n,m)=2\tan^{-1}\left(\frac{x_0}{\sqrt{2\dfrac{n^2\pi^2}{a^2}+x_0^2}}\right)+bx_0+m\pi=0,\quad m\in\mathbb{Z},\ n\in\mathbb{Z}^+. \tag{21}
$$

4.2. Case 2: S–C–S–F

Setting the ε^0 term in Eq. (18) to zero, with $z_0=z(r_n,x_0)$ given by Eq. (17), we have

$$
0=\mathrm{e}^{\mathrm{i}bx_0}z_0-\mathrm{e}^{-\mathrm{i}bx_0}\overline{z_0}
$$

or

$$
\begin{aligned}
2bx_0+2\pi m &= \arg\left(\frac{\overline{z_0}}{z_0}\right)\\
&= 2\arg(\overline{z_0})\\
&= 2\arg\Big\{x_0^2f_2^2(r_n,x_0)-(2r_n^2+x_0^2)f_1^2(r_n,x_0)\\
&\qquad -\mathrm{i}x_0\sqrt{2r_n^2+x_0^2}\,\left[f_1^2(r_n,x_0)+f_2^2(r_n,x_0)\right]\Big\}\\
&= 2\tan^{-1}\left(\frac{x_0\sqrt{2r_n^2+x_0^2}\,\left[f_1^2(r_n,x_0)+f_2^2(r_n,x_0)\right]}{(2r_n^2+x_0^2)f_1^2(r_n,x_0)-x_0^2f_2^2(r_n,x_0)}\right).
\end{aligned}
$$

So we must solve

$$F(x_0,n,m) = \tan^{-1}\left(\frac{x_0\sqrt{2r_n^2+x_0^2}\,\left[f_1^2(r_n,x_0)+f_2^2(r_n,x_0)\right]}{(2r_n^2+x_0^2)f_1^2(r_n,x_0)-x_0^2f_2^2(r_n,x_0)}\right) - bx_0 - m\pi = 0, \quad m\in\mathbb{Z},\ n\in\mathbb{Z}^+. \tag{22}$$

4.3. Case 3: S–F–S–F

Here the ε^0 term in Eq. (20) is set to zero and z_0 is given by Eq. (19). Thus, we have

$$\begin{aligned} 2bx_0 + 2\pi m &= 2\arg(\overline{z_0}) \\ &= 2\arg\Big[f_1^4(r_n,x_0)(2r_n^2+x_0^2) - x_0^2f_2^4(r_n,x_0) \\ &\quad + 2\mathrm{i}x_0\sqrt{2r_n^2+x_0^2}f_1^2(r_n,x_0)f_2^2(r_n,x_0)\Big] \\ &= 2\arg\left[f_1^2(r_n,x_0)\sqrt{2r_n^2+x_0^2} + \mathrm{i}x_0f_2^2(r_n,x_0)\right]^2 \\ &= 4\tan^{-1}\left(\frac{x_0f_2^2(r_n,x_0)}{f_1^2(r_n,x_0)\sqrt{2r_n^2+x_0^2}}\right). \end{aligned}$$

So, we solve

$$F(x_0,n,m) = 2\tan^{-1}\left(\frac{x_0f_2^2(r_n,x_0)}{f_1^2(r_n,x_0)\sqrt{2r_n^2+x_0^2}}\right) - bx_0 - m\pi = 0, \quad m\in\mathbb{Z},\ n\in\mathbb{Z}^+. \tag{23}$$

Now, in the literature, we have found numerous data for the cases S–C–S–C and S–F–S–F, but only for the first few frequencies. Meanwhile, we have not found any data for S–C–S–F. Thus, we have also written a Legendre-tau spectral approximation [7] to the problem. The algorithm is, in principle, the same as that described in Ref. [5] (though in practice, noticeably more complex), so we provide only a brief description here.

The Legendre-tau approximation entails transforming PDE (11) and the boundary conditions to a problem on the square $[-1,1]\times[-1,1]$, where we then expand the unknown function $\phi(x,y)$ in the form

$$\phi(x,y) = \sum_{n=0}^{N}\sum_{m=0}^{M} a_{nm}P_n(x)P_m(y).$$

Here, P_n is the Legendre polynomial of degree n. Substituting the sum into the PDE and BCs allows us to form an $(N+1)(M+1)\times(N+1)(M+1)$ linear system in the unknowns a_{nm} and λ. Thus, we are led to a generalized matrix eigenvalue problem in the frequency λ.

5. Results and comparisons

In each case we have considered the square plate with $a=b=1$. The 0th-order equations (21)–(23) were solved using a standard Newton's method, and in these three cases the worst error is $|F(x_0,n,m)|\approx 10^{-8}$, $|F(x_0,n,m)|\approx 10^{-7}$ and $|F(x_0,n,m)|\approx 10^{-9}$, respectively.

The generalized eigenvalues of the Legendre-tau matrix were computed using the IMSL routine DEVLRG [8]. In each case, we have used $N=M=18$ Legendre polynomials in each independent variable.

Computations for $N = M = 20$ show all frequencies appearing in the tables to have converged to at least seven decimal places. All computations were performed on the DEC Alpha 2100 at Fairfield University.

5.1. Case 1: S–C–S–C

As we have set the coefficient of the biharmonic operator in Eqs. (1) and (11) to be unity, Poisson's ratio ν does not play an explicit role unless at least one BC is F. Table 1 gives the first five frequencies, after which the 0th-order perturbation results match those from Legendre-tau. The first four columns give the frequencies $\lambda^{(m)} = (n\pi)^2 + (x^{(m)})^2$, for appropriate n, resulting from the perturbation calculation. The column labelled L-T gives the Legendre-tau computations while the column labelled B lists those values which have been computed by Bardell [9], using the hierarchical finite-element method. Note that Bardell's paper also gives the values computed by Leissa [1] and Liew et al. [10], all computed numerically, and all of which are near to exact matches with Bardell's.

In this table, and in Tables 2 and 3, we have carried through the perturbation process until the results show no further improvement or until they match the Legendre-tau results.

Table 1
Comparison of the first five frequencies for the 1 × 1 S–C–S–C plate

$\lambda^{(0)}$	$\lambda^{(1)}$	$\lambda^{(2)}$	L-T	B
28.9145	28.9509	28.9508	28.9508	28.95
54.7423	54.7437		54.7431	54.74
69.3313	69.3270		69.3270	69.33
94.5856	94.5853		94.5853	94.59
102.216			102.216	

Table 2
Comparison of the first seven frequencies for the 1 × 1 S–C–S–F plate, for $\nu = 0.25$

$\lambda^{(0)}$	$\lambda^{(1)}$	$\lambda^{(2)}$	L-T
12.8302	12.8614	12.8614	12.8618
33.3410	33.3189	33.3188	33.3174
41.9575	41.9576		41.9578
63.4349	63.4345		63.4345
72.6019	72.6052		72.6054
90.9709	90.9708	90.9707	90.9707
103.670			103.670

Table 3
Comparison of the first six frequencies for the 1 × 1 S–F–S–F plate, for $\nu = 0.3$

$\lambda^{(0)}$	$\lambda^{(1)}$	$\lambda^{(2)}$	$\lambda^{(3)}$	L-T	B
9.86960	9.86960*			9.62641	9.63
16.1659	16.1350	16.1349	16.1348	16.1323	16.13
36.7071	36.7257	36.7256		36.7255	36.73
39.4784	39.4784*			38.9448	38.95
70.7392	70.7401			70.7401	
75.2824				75.2824	

Those frequencies marked "*" show no improvement.

5.2. Case 2: S–C–S–F

Here Poisson's ratio does appear explicitly in the free boundary condition. We take a typical value of $v = 0.25$ for our computations. Table 2 is set up exactly as Table 1, except that there are no results in the literature with which to compare. Also, we need to go down to the 7th frequency before the 0th-order and Legendre-tau results match.

5.3. Case 3: S–F–S–F

In this case, we use $v = 0.3$ as that is the value used in all relevant references in the literature. Table 3 is set up exactly as Table 1. Here we include the first six frequencies, until the 0th-order and Legendre-tau results match. However, the asterisked entries jump out, as they show no improvement. The problem here is that the actual 0th-order solution is $x_0 = 0$. Thus, as the Newton's method gives us a value $x_0 \approx 0$, the ε-coefficient in Eq. (20) then gives us $x_1 \approx 0$. (The same thing happens if we try to consider ε terms of higher power.) Note that we cannot have $k_2 \neq 0$ because the problem has no solution of the form $w(x, y, t) = g(x, t)$, due to the presence of at least one F boundary condition.

Thus, the perturbation approach fails for those configurations which have zero as a 0th-order, y-direction wave number, and it is not hard to see that this occurs exactly when the edges $y =$ constant are R-opposite-F and F-opposite-F (and also R-opposite-R, but that case can be solved exactly).

We will solve this problem in the following section, then in Table 5 we provide a list of the first 20 frequencies for each of the three configurations. We do so in order to provide benchmarks for those not appearing elsewhere in the literature.

Before moving on, it is instructive to compare these plate problems with the "corresponding" beam problems. Specifically, we compare the S–C–S–C plate with the C–C Euler–Bernoulli beam, S–C–S–F with C–F, and S–F–S–F with F–F. First, it is easily seen that, if we let $k_1 = 0$, each plate matrix M_ε becomes identical to the corresponding beam matrix (C–C and C–F can be found in Ref. [2]). Now $k_1 = 0$ is not a solution to the plate problem with S-opposite-S or S-opposite-R, but it *is* when we have R-opposite-R.

It is also interesting to compare them asymptotically. By letting $x_0 \to \infty$ in the 0th-order equations (21)–(23) we get, in each case,

$$x_0 \sim \frac{2m+1}{2}\pi, \quad m \in \mathbb{Z},$$

which is the same as for the corresponding beams. (Again, C–C and C–F are given in Ref. [2], while F–F is easy, as well.)

6. The case when zero is a 0th-order solution

The perturbation method fails in this case and there seems to be no way to modify it so that it will work. However, led by the perturbation results for other frequencies, we are in a position where we can solve for the troublesome frequencies "exactly" (similar to what is done in Ref. [11]). Specifically, the least positive value for $x = k_2$ for the configuration is $k_2 = 2.50305$, so we have a nice bounded interval on which to search for the unknown k_2. How?

For those cases when k_2 is near zero, we expect the solutions to be "almost constant" along the lines $x =$ constant, that is, we expect the solutions not to be sinusoidal in the y-direction. So we form, instead of Eq. (12), the "general solution"

$$\phi(x,y) = A_1 e^{-in\pi x + k_2 y} + A_2 e^{in\pi x + k_2 y} + A_3 e^{in\pi x - k_2 y} + A_4 e^{-in\pi x - k_2 y} + B_1 e^{-\sqrt{n^2\pi^2 - 2k_2^2}x + k_2 y} + B_2 e^{-in\pi x - \sqrt{2n^2\pi^2 - k_2^2}(1-y)}$$

Table 4
Results of solving Eq. (24) using the bisection method

n	k_2	$\lambda = n^2\pi^2 + k_2^2$	L-T	B
1	0.493303	9.62626	9.62641	9.63
2	0.730483	38.9448	38.9448	
3	0.916378	87.9867	87.9867	
4	1.07758	156.753	156.753	

Table 5
The first 20 frequencies for the S–C–S–C, S–C–S–F and S–F–S–F plates

S–C–S–C	S–C–S–F ($v = 0.25$)	S–F–S–F ($v = 0.3$)
28.9508	12.8618	9.62641
54.7431	33.3174	16.1323
69.3270	41.9578	36.7255
94.5853	63.4345	38.9448
102.216	72.6054	70.7401
129.096	90.9707	75.2824
140.205	103.670	87.9867
154.776	112.381	96.0672
170.346	131.592	111.025
199.811	153.435	122.040
206.697	162.867	133.700
208.392	180.974	156.752
234.585	210.391	164.695
258.614	213.415	164.866
265.196	222.271	169.536
279.651	241.577	191.871
293.756	248.354	212.099
307.316	261.901	224.723
333.953	269.358	236.262
344.538	282.892	245.242

$$
\begin{aligned}
&+ B_3 e^{-\sqrt{n^2\pi^2-2k_2^2}(1-x)+k_2 y} + B_4 e^{\mathrm{i}n\pi x-\sqrt{2n^2\pi^2-k_2^2}y} \\
&+ C_1 e^{-\sqrt{n^2\pi^2-2k_2^2}x-k_2 y} + C_2 e^{-\mathrm{i}n\pi x-\sqrt{2n^2\pi^2-k_2^2}y} \\
&+ C_3 e^{-\sqrt{n^2\pi^2-2k_2^2}(1-x)-k_2 y} + C_4 e^{\mathrm{i}n\pi x-\sqrt{2n^2\pi^2-k_2^2}(1-y)}.
\end{aligned}
$$

In fact, this is just Eq. (12) with $k_1 = n\pi,\ a = b = 1$ and, interestingly, with k_2 replaced by $\mathrm{i}k_2$.

Now we can proceed as before and apply the boundary conditions, in this case S–F–S–F. After all is said and done, we are left to solve the equation

$$
0 = e^{x}\left[\sqrt{2n^2\pi^2 - x^2} f_1^2(n\pi, x) - x f_2^2(n\pi, x)\right]^2 - e^{-x}\left[\sqrt{2n^2\pi^2 - x^2} f_1^2(n\pi, x) + x f_2^2(n\pi, x)\right]^2 \tag{24}
$$

on the closed interval $[0, 2.50305]$. Here, again, f_1 and f_2 are given in Eq. (16). This problem is quite easy to solve using a Newton's method or even the bisection method. We have done so, using the latter, and the results are presented in Table 4. Again, L-T and B are the Legendre-tau and Bardell results, respectively, and we also list the value of k_2 in each case.

We end with Table 5, in which we give benchmark data by listing the first 20 frequencies for each of the S–C–S–C, S–C–S–F and S–F–S–F plates. Where the asymptotic and Legendre-tau results do not match exactly, we have used the latter. It should be noted, however, that for high enough frequencies, the Kirchhoff–Love conditions are violated. Thus, results for the higher vibration modes may be unreliable.

References

[1] A.W. Leissa, The free vibration of rectangular plates, *Journal of Sound and Vibration* 31 (3) (1973) 257–293.

[2] G. Chen, M.P. Coleman, Improving low order eigenfrequency estimates derived from the wave propagation method for an Euler–Bernoulli beam, *Journal of Sound and Vibration* 204 (4) (1997) 696–704.

[3] M.P. Coleman, Vibration eigenfrequency analysis of a single-link flexible manipulator, *Journal of Sound and Vibration* 212 (1) (1998) 109–120.

[4] J.B. Keller, S.I. Rubinow, Asymptotic solution of eigenvalue problems, *Annals of Physics* 9 (1960) 24–75.

[5] G. Chen, M.P. Coleman, J. Zhou, Analysis of vibration eigenfrequencies of a thin plate by Keller–Rubinow's wave method (I): clamped boundary conditions with rectangular or circular geometry, *SIAM Journal of Applied Mathematics* 51 (1991) 967–983.

[6] V.V. Bolotin, An asymptotic method for the study of the problem of eigenvalues for rectangular regions, in: J.R.M. Radds (Ed.), *Problems of Continuum Mechanics*, SIAM, Philadelphia, 1961, pp. 56–68.

[7] D. Gottlieb, S.G. Orszag, *Numerical Analysis of Spectral Methods, Theory and Applications*, SIAM, Philadelphia, 1977.

[8] *IMSL Math/Library*: *Fortran Subroutines for Mathematical Applications*, IMSL, Houston, TX, 1987.

[9] N.S. Bardell, Free vibration analysis of a flat plate using the hierarchical finite element method, *Journal of Sound and Vibration* 151 (2) (1991) 263–289.

[10] K.M. Liew, K.Y. Lam, S.T. Chow, Free vibration analysis of rectangular plates using orthogonal plate functions, *Computers & Structures* 34 (1) (1990) 79–85.

[11] M.A. Morales, S. Botello, R. Salinas, A root finding technique to compute eigenfrequencies for elastic beams, preprint.